JOURNAL OF COMPUTATIONAL PHYSICS 58, 29–43 (1985)

Accuracy of the Random Vortex Method for a Problem with Non-smooth Initial Conditions*

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Received June 22, 1983; revised November 9, 1983

The accuracy of the random vortex method, in approximating the solution of the Navier–Stokes equation, is investigated using the model problem of a circular vortex as suggested by Milinazzo and Saffman [13]. The method consists of partitioning the vorticity into "vortex blobs." These blobs are moved via two actions. First, a blob is deterministically moved under the action of the velocity field associated with the other blobs. Then to simulate viscosity a random component is added to the position of the blob. For this model problem the nonlinear terms of the Navier–Stokes equation vanish. Thus the major error inherent in the deterministic component of the method vanishes. Consequently, for this model problem concentration is on the interaction of the deterministic and random components of the method. Results show that the accuracy of the method depends heavily on the initial distribution and strength of the computational elements, i.e., the vortex blobs. With the right choice of initial conditions it is found that $e_1(t) = |L(t) - A(t)|/A(t)$ is $O(R^{-1/2}N^{-1/2})$, where $L(t)$ and $A(t)$ are, respectively, the exact and computed angular moment of vorticity distribution at time t for N vortex blobs at a Reynolds number R.

1. INTRODUCTION

The vorticity stream formulation of the incompressible Navier–Stokes equation in the plane has the form,

$$\omega_t + u\omega_z + v\omega_y = \frac{1}{R}\Delta\omega, \qquad \omega(\bar{z}, 0) = \xi(\bar{z}) \tag{1.1}$$

$$\Delta\psi = \omega \tag{1.2}$$

$$u = -\psi_y \tag{1.3}$$

$$v = \psi_x, \tag{1.4}$$

where $\bar{u} = (u, v)$ is the velocity, ω the vorticity, ψ the stream function, R is the Reynolds number, and $\bar{z} = (x, y)$ are the spatial coordinates.

* This work was supported in part by the Director, Office of Energy Research, Office of Basic Energy Sciences, Engineering, Mathematical, and Geosciences Division of the U.S. Department of Energy, under Contract DE-AC03-76SF00098.

0021-9991/85 $3.00

To solve these equations numerically it is common to solve a discrete approximation of these equations on a grid. These approximations introduce an artifical viscosity term. The random vortex method (Chorin [5]) is designed to overcome this difficulty. The method involves partitioning the vorticity into a sum of "blobs." It is a fractional step method. At each time step, the convective part of the Navier–Stokes equation is solved by moving each of the blobs under the action of the velocity field associated with the other blobs (vortex method). The diffusion part is simulated by letting the particles undertake a random walk (Lamperti [10, Chap. 4]). It should be noted that large statistical errors can occur in the solution of vorticity by using this method. On the other hand, as velocity is an integral of vorticity, the statistical error for the velocity is much smaller. Chorin's method also allows for domains with boundaries. We shall use this method in the case in which the domain is the whole plane. Thus it is not necessary to discuss the boundary conditions.

Chorin's method has, on the whole (see Leonard [11]), been received as a useful and important tool in the study of high Reynolds number, incompressible flows. However, Milinazzo and Saffman [13] came to the conclusion that the method could produce significant errors, even when a large number of vortex blobs were used. They obtained 10% error with 1000 blobs for a Reynolds number of 5000. Chorin [6] has pointed out that their conclusions were based on their choice of error measurement. He also criticizes their implementation of the random vortex method, in particular their choice of cutoff parameter (see Sect. 2).

It is our intent to reproduce Milinazzo and Saffman's results, in order to clarify their results and their disagreement with Chorin. In Section 3, we present Milinazzo and Saffman's model problem. We analyze two methods of measuring the error for the method, the one proposed by Chorin in [6], and the one used by Milinazzo and Saffman in their paper. Estimates are derived for the errors expected in the numerical results. These estimates are based on the calculations of Milinazzo and Saffman [13], Section 4, and agree with Chorin's [5] estimates.

In Section 4 we compare the method used by Chorin and others to approximate the initial vorticity and the construction of Milinazzo and Saffman. In the first method the vortex blobs are initially placed on a uniform grid. In the second method the blobs are placed by use of a random distribution. We find that the first method produces more accurate results. For instance, using 500 blobs and a Reynolds number of 5000, we obtain a 5% error using the first method and a 12% error using the second method. These errors are calculated using Milinazzo and Saffman's method of error measurement. This result verifies Milinazzo and Saffman's result of a 10% error for 1000 blobs, since an increase in the number of blobs by a factor of 2 should decrease the error by a factor of $\sqrt{2}$. Note that if we use Chorin's error measurement for the two cases above, we obtain errors of 0.3% and 0.7%, respectively.

We find that our results are insensitive to both the type of cutoff function used and the size of cutoff employed. In particular, we tested the cutoffs used by Beale and Majda [4], Chorin [5], Hald [7], and Milinazzo and Saffman [13] (see

Table I). Since the average distribution of particles for our problem is radially symmetric it is reasonable to assume that the angular motion of the vortex particles is governed by the vortex method, whereas the radial motion is governed by the random walk algorithm. As our methods of error measurement are only sensitive to the average change in the radial position of the particles, it is not surprising that we cannot detect any difference between the various cutoff functions.

Our numerical runs show that there are two types of errors in the computations. We refer to these as the startup error and the interaction error. The startup error depends on the difference between the initial distribution of vortex blobs and the exact vorticity. The interaction error involves the errors produced through the interaction of the vortex method and the random walk algorithms at each time step. Our computations show that Milinazzo and Saffman's implementation gives 2 to 5 times larger startup errors than the standard method in which the vortex blobs are initially placed on a uniform grid. For a Reynolds number greater than 1000 the startup error "swamps" the interaction error, i.e., the computed error is due to Milinazzo and Saffman's choice of initial conditions. However, for larger Reynolds numbers ($R \geqslant 20000$) even the standard method produces significant startup errors in comparison to the interaction error.

TABLE I

Common Cutoff Functions

Chorin [5]

$$f_\delta(r) = 1 \quad \text{if} \quad r > \delta,$$
$$= r/\delta \quad \text{if} \quad r < \delta$$

Milinazzo and Saffman [13]

$$f_\delta(r) = 1 \quad \text{if} \quad r > \delta,$$
$$= r^2/\delta^2 \quad \text{if} \quad r < \delta$$

Hald [7]

$$f_\delta(r) = \frac{r^2}{\delta^2}\left[14 - 105\frac{r^2}{\delta^2} + 196\frac{r^3}{\delta^3} - 140\frac{r^4}{\delta^4} + 36\frac{r^5}{\delta^5}\right] \quad \text{if} \quad r < \delta, 1 \text{ if } r > \delta$$

$$f_\delta(r) = \frac{r^2}{3\delta^2}\left[56 - 630\frac{r^2}{d^2} + 1568\frac{r^3}{\delta^3} - 1680\frac{r^4}{\delta^4} + 864\frac{r^5}{\delta^5} - 175\frac{r^6}{\delta^6}\right] \quad \text{if} \quad r < \delta, 1 \text{ if } r > \delta$$

Beale and Majda [4]

$$f_\delta(r) = 1 - e^{-r^2/\delta^2}$$

$$f_\delta(r) = 1 - 2e^{-r^2/\delta^2} + e^{-r^2/2\delta^2}$$

$$f_\delta(r) = 1 - \left(\frac{r^2}{\delta^2} - 1\right)e^{-r^2/\delta^2}$$

By measuring the error in a different way we can study the interaction term. We observe that the errors for the random vortex method and the solution of the diffusion equation using the random walk method are comparable, at least if the number of vortices are less than 500. Both methods produce errors of $O(N^{-1/2}R^{-1/2})$, where N is the number of blobs. This verifies our analysis in Section 3. The error for the random vortex though, is typically larger than that of the random walk algorithm.

2. The Random Vortex Method

The random vortex method solves the Navier–Stokes equation (1.1)–(1.4) by a fractional step method, which consists of the two equations,

$$\omega_t = Q\omega, \qquad \omega(\bar{z}, 0) = \xi_1(\bar{z}) \tag{2.1}$$
$$Q\omega = -(u\omega_z + v\omega_y),$$

where

$$\Delta\psi = \omega, \qquad u = -\psi_y, \qquad v = \psi_z$$

and

$$\omega_t = \frac{1}{R}\Delta\omega, \qquad \omega(\bar{z}, 0) = \xi_2(\bar{z}). \tag{2.2}$$

At each time step, Eq. (2.1) is solved by following the evolution of a finite number of fluid elements (vortex blobs). The solution of Eq. (2.2) is then simulated by adding a gaussianly distributed random component, of appropriate variance, to the position of these fluid elements (blobs). An approximation to the Navier–Stokes equation is then obtained from the resulting vorticity distribution.

We will discuss the solution of Eq. (2.1) by the vortex method (Chorin [5]). The basic idea is to approximate the vorticity by a sum of "vortex blobs." A blob can be thought of as a region of vorticity which is convected in the fluid under the action of the other blobs. In a real fluid, the distribution of vorticity inside a blob would be distorted by the flow. In the vortex method it is assumed that the blobs are translated as a whole. The validity of this is proved in Hald [7] and Beale and Madja [4]. The problem is then reduced to following the evolution of a finite number of these blobs.

First it is necessary to choose a set of "cutoff functions," $\{\varphi_\delta\}_{\delta>0}$. These functions are approximate δ functions on the plane, such that for every function, $u \in L^1 \cap L^\infty$,

$$\varphi_\delta * u \to u \text{ as } \delta \to 0, \tag{2.3}$$

Here $*$ denotes convolution. We call δ the "cutoff" for the function φ_δ. For any

function φ with $\int \varphi = 1$, $\varphi_\delta(\bar{z}) = \delta^{-2}\varphi(\bar{z}/\delta)$ is a set of cutoff functions. For $\omega \in L^1 \cap L^\infty$, and small δ we have the following approximation:

$$\omega(\bar{z}) \approx \varphi_\delta * \omega(\bar{z}) \approx \sum_i \gamma_i \varphi_\delta(\bar{z} - \bar{z}_i) \tag{2.4}$$

for some points $\bar{z}_i$ and constants γ_i. The first approximation in (2.4) is due to relation (2.3), the second approximation is simply a standard Riemann sum approximation to the convolution integral. We can chose the parameters $\bar{z}_i$ and γ_i in various ways. One approach is to allow the points $\bar{z}_i$ to lie initially on uniform grid of mesh size h, and to let $\gamma_i = h^2\omega(\bar{z}_i)$; another is to let $\gamma_i = \int_{B_i} \omega$, where B_i denotes the square centered on $\bar{z}_i$ of area h^2.

The "vortex blobs" or simply the "blobs" are represented by the vorticity distributions $\omega_i(\bar{z}) = \gamma_i \varphi_\delta(\bar{z} - \bar{z}_i)$ (here we have suppressed the δ dependence of ω). We say that the center of the ith blob is $\bar{z}_i$ and that is strength is γ_i.

Let G be the fundamental solution to Laplace's equation and let G_δ denote the smoothed kernel $G * \varphi_\delta$. Then we have by (2.1) and (2.4) that the stream function ψ satisfies the relation

$$\psi(\bar{z}) = G * \omega(\bar{z}) \approx \sum_i \gamma_i G * \varphi_\delta(\bar{z} - \bar{z}_i) = \sum_i \gamma_i G_\delta(\bar{z} - \bar{z}_i). \tag{2.5}$$

Thus an approximate velocity field is given by

$$u(\bar{z}) = -\psi_y = -\sum_i \gamma_i \frac{\partial G_\delta}{\partial y}(\bar{z} - \bar{z}_i) \tag{2.6}$$

$$v(\bar{z}) = \psi_x = \sum_i \gamma_i \frac{\partial G_\delta}{\partial x}(\bar{z} - \bar{z}_i). \tag{2.7}$$

We use the equations to update the positions of the vortex blobs. Specifically, we suppose that the positions $\bar{z}_i(t) = (x_i(t), y_i(t))$ of the blobs satisfy the first-order system of differential equations

$$\frac{dx_j}{dt} = -\sum_i \gamma_i \frac{\partial G_\delta}{\partial y}(\bar{z}_j - \bar{z}_i)) \tag{2.8}$$

$$\frac{dy_j}{dt} = \sum_i \gamma_i \frac{\partial G_\delta}{\partial x}(\bar{z}_j - \bar{z}_i)). \tag{2.9}$$

If the cutoff functions have radial symmetry we introduce the functions $K(\bar{z})$ and $K_\delta(\bar{z})$ and the associated function $f_\delta(r)$ defined by

$$K(\bar{z}) = (-\partial_y, \partial_x)\, G(\bar{z}) = \frac{1}{2\pi |\bar{z}|^2}(-y, x) \tag{2.10}$$

$$K_\delta(\bar{z}) = (-\partial_y, \partial_x)\, G_\delta(\bar{z}) = f_\delta(|\bar{z}|)\, K(\bar{z}). \tag{2.11}$$

In Table I, we have displayed the functions f_δ associated with some of the more common choices of cutoff functions. Notice that $f_\delta \to 1$ as $\delta \to 0$ or as $r \to \infty$.

Using the functions defined above we can also rewrite Eqs. (2.6) and (2.7) in the simpler form

$$(u(\bar{z}), v(\bar{z})) = \sum_i \gamma_i K_\delta(\bar{z} - \bar{z}_i). \tag{2.12}$$

The convergence of the vortex method for smooth initial data, using specific cutoff functions, has been proved. These results state that h, the initial distance between the blobs, and the cutoff δ, must satisfy a relation of the form $\delta = h^q$, where $0 < q < 1$. Hald and Del Prete [9] first showed convergence using their cutoffs. Hald [7] subsequently improved those estimates. Finally Beale and Majda [4] have shown that there exist cutoff functions which can produce arbitrary orders of convergence.

The solution of Eq. (2.2) is performed by the following random walk method. It is well known that the probability distribution of the positions of particles undergoing a Brownian motion satisfies Eq. (2.2), the diffusion equation (see Lamperti [10]). The idea is to use a gaussianly distributed sequence to add a random component to the positions of the vortex blobs at each time step. This will simulate the diffusion inherent in Eq. (2.2).

An exact analysis of the random vortex method is not available. Marchioro and Pulvirenti [12] have shown that the random vortex method approximates the Navier–Stokes equations in a weak sense, as the initial grid size and cutoff approaches 0 in an appropriate way. Their result is impractical for us as we need stronger forms of convergence. For instance, we need convergence in an L^2 or L^∞ sense. Hald [8] has analysed a method similar to the random vortex method for a problem in chemical dynamics and obtained strong convergence. His results and the discussions found in [5, 13, 3] indicate that random vortex method will approximate the solution of the Navier–Stokes equation with an error of order $R^{-1/2}$. This is of the same order as the error obtained when Eq. (2.2) is solved by using random walks.

3. The Model Problem

Our model problem consists of solving Eqs. (1.1)–(1.4), together with the non-smooth initial condition

$$\begin{aligned} \xi(\bar{z}) &= 1 \quad \text{if} \quad |\bar{z}| \leqslant r, \\ &= 0 \quad \text{if} \quad |\bar{z}| \geqslant r. \end{aligned} \tag{3.1}$$

As this is radially symmetric, it follows that $Q\omega = 0$ (see Batchelor [1]). The Navier–Stokes equations reduce to two cases,

$$\omega_t = 0 \qquad \text{if} \quad R = \infty, \tag{3.2}$$

$$\omega_t = \frac{1}{R}\Delta\omega \qquad \text{if} \quad 0 < R < \infty. \tag{3.3}$$

If $R = \infty$, the solution is a rotating blob with angular velocity $\frac{1}{2}$, and period of rotation $T = 4\pi$. For $0 < R < \infty$, ω is a solution of the heat equation and can be written in the form,

$$\omega(\bar{z}, t) = \int_{\mathbb{R}^2} E_t(\bar{z} - \bar{z}_1)\, \xi(\bar{z}_1)\, d\bar{z}_1, \tag{3.4}$$

where ξ is the initial distribution of vorticity and E_t is the fundamental solution for the heat equation

$$E_t(\bar{z}) = \frac{R}{4\pi t} e^{-(R|\bar{z}|^2/4t)}. \tag{3.5}$$

One test of our numerical method is to calculate the exact and numerical values of vorticity (or velocity) at many points and use a discrte norm (e.g., L^2) to measure the error of the method. Another is to compare the positions of the vortex blobs at each time step with the positions of corresponding particles evolved by the exact flow. However, it is expensive to evaluate the integral (3.4) or to calculate the vorticity generated by the vortex blobs at many points. A common practice in these situations is to compare the exact and computed values of easily computable functionals. Following Milinazzo and Saffman, we chose the functional

$$L(t) = \frac{\int_{\mathbb{R}^2} |\bar{z}|^2\, \omega(\bar{z}, t)\, d\bar{z}}{\int_{\mathbb{R}^2} \omega(\bar{z}, t)\, d\bar{z}} \tag{3.6a}$$

which satisfies

$$L(t) = L(0) + 4t/R. \tag{3.6b}$$

In the random vortex method L can be approximated by

$$A(t) = \frac{1}{N}\sum_i [(x_i(t))^2 + (y_i(t))^2], \tag{3.7}$$

where the number of vortex blobs is denoted by N and their centers by $(x_i(t), y_i(t))$. The derivation of this formula supposes that all of the vortex blobs in our test problem have equal strength.

We will study the expected value denoted by E, and the variance denoted by σ^2, of the quantity A. An exact analysis is not available, but we expect that the general case should be comparable to pure diffusion. It follows from Section 4 of Milinazzo and Saffman [13] that,

$$E(A(t)) = A(0) + 4t/R \tag{3.8}$$

$$\sigma^2(A(t)) = \frac{8t}{RN} A(0) + \frac{8t^2}{NR^2}\left[5 - \frac{3\Delta t}{t}\right]$$

$$\approx \frac{8t}{RN} A(0) \qquad \text{for} \quad \frac{R}{t} \gg 1. \tag{3.9}$$

The analysis of the error is based on the functional L and its numerical approximation A. We considered the relative error in A,

$$e(t) = \frac{|A(t) - L(t)|}{|L(t)|}. \tag{3.10}$$

Thus if the error in vorticity is small, then $e(t)$ will be small. The function $e(t)$ can be decomposed into two components corresponding to "startup" and "interaction" errors. Thus we introduce the functions

$$e_{\text{startup}} = \frac{|A(0) - L(0)|}{|L(0)|}, \tag{3.11}$$

$$e_1(t) = \frac{|A(t) - [A(0) + 4t/R]|}{|A(t)|}. \tag{3.12}$$

Note that $A(0) + 4t/R$ is the expected value of $A(t)$.

In the case of pure diffusion the standard deviation $\sigma(e_1(t))$ can be calculated from (3.9) and satisfies

$$\sigma(e_1(t)) \approx \frac{1}{N^{1/2}R^{1/2}}\left[\frac{8t}{A(0)}\right]^{1/2} \qquad \text{for} \quad R/t \gg 1. \tag{3.13}$$

Milinazzo and Saffman [13] measure the error by use of the functional

$$e_{\text{MS}}(t) = \frac{|A(t) - [L(0) + 4t/R]|}{4t/R}. \tag{3.14}$$

It is possible to construct examples in which the vorticity field generated by a sum of vortex blobs approximates the exact vorticity for the model problem arbitrarily well, but for which the function $e_{\text{MS}}(t)$ is of arbitrary size (in particular, when R is large or t small). Thus we prefer to estimate the error by e_{startup} and e_1.

4. Numerical Results

There are many possible ways to implement the random vortex method. For example, we can make the following choices:

(1) Initial placement and strength of the vortex blobs.

(2) Form of cutoff function and the appropriate cutoff size for a given number of blobs.

(3) Choice of time integration scheme to solve the system of Eqs. (2.8) and (2.9).

(4) Random number generator to be used for the simulation of the random walk.

Our implementation consisted of the following choices:

The positions $\bar{z}_i$ of the blobs were initially given on a uniform grid of mesh size h. Let B_i be the square of area h^2 centered on $\bar{z}_i$ and parallel to the grid. Then the strengths of each blob are given by, $\gamma_i = \int_{B_i} \omega$. We call this a uniform grid distribution with averaging.

For the cutoff function we chose Chorin's (see Table I) with a cutoff $\delta = 2h$.

To integrate Eqs. (2.8) and (2.9), we used fourth-order Runge–Kutta as our time integration method with a time step of $\Delta t = 0.05$. With this choice, the vortex method (without random walk) produced a change in the value of $A(t)$ which was less than 10^{-15}, over a time interval of 3 periods of rotation of exact solution. Milinazzo and Saffman [13] chose Heun's method, whereas Chorin [5] used Euler's method.

To simulate the random walk we needed a 2-dimensional gaussianly distributed

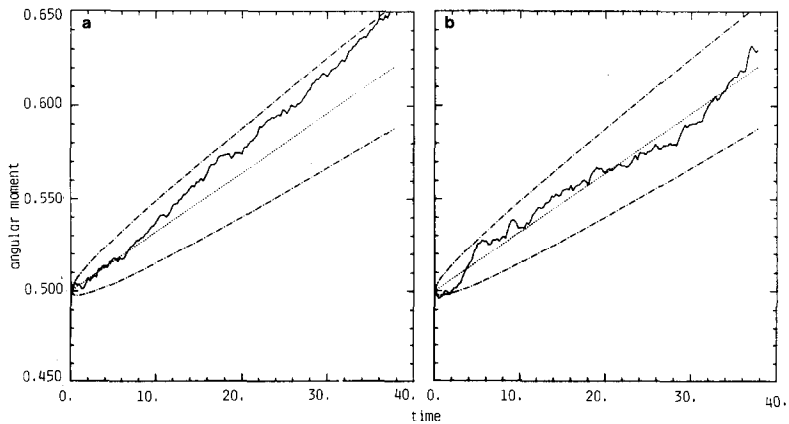

FIG. 1. (a) Angular moment for 429 blobs evolved by random walk algorithm for $R = 1250$ together with its expected value and standard deviation. (b) Angular moment for 468 blobs evolved by random walk algorithm for $R = 1250$ together with its expected value and standard deviation.

sequence of points (ζ_1, ζ_2), with zero mean and standard deviation v. This was obtained from a uniformly distributed sequence of points (η_1, η_2), on $[0, 1] \times [0, 1]$, by using the formula

$$\zeta_1 = v \cos(2\pi\eta_1)(-2\log(\eta_2))^{1/2} \tag{4.1}$$

$$\zeta_2 = v \sin(2\pi\eta_1)(-2\log(\eta_2))^{1/2}. \tag{4.2}$$

Figures 1a, b and 2a, b display typical results. These figures display the computed value of $A(t)$, compared to the expected value of $A(t)$ and its standard deviation as given by (3.8) and (3.9) ($R = 1250$, $N = 429$ and 468, $\Delta t = 0.05$). Notice that the errors demonstrated by the random vortex method, for this relatively small Reynolds number, are qualitatively similar to the corresponding results for the random walk algorithm.

We studied the effect of varying the initial distribution of the vortex blobs and the choice of cutoff function. Following Milinazzo and Saffman [13, p. 384], we investigated the effect of distributing the blobs randomly at $t = 0$. The positions $\bar{z}_i$ of the blobs were generated by using a uniformly distributed sequence of points on $[0, 1] \times [0, 1]$. This type of initial distribution will be called random, and h will denote the average distance between the blobs. The strengths of the blobs were determined by either:

(1) $\gamma_i = \int_{B_i} \omega$, where B_i denotes the square of area h^2 centered on $\bar{z}_i$;

(2) $\gamma_i = \omega(\bar{z}_i)\, h^2$.

Figures 3a–d display the errors obtained for the various initial startup procedures discussed. Notice the particularly "choppy" behavior of Figs. 3c, d. This can be explained by observing that the addition of a blob at position (x, y) will change the

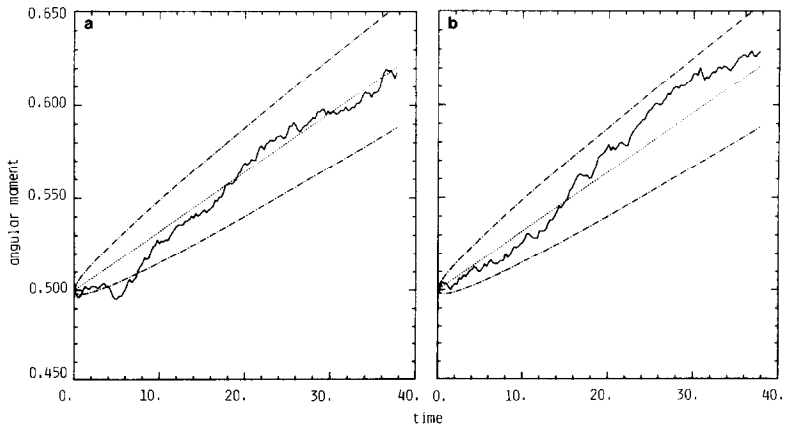

FIG. 2. (a) Angular moment for 429 blobs evolved by random vortex method for $R = 1250$ together with its expected value and standard deviation. (b) Angular moment for 468 blobs evolved by random vortex method for $R = 1250$ together with its expected value and standard deviation.

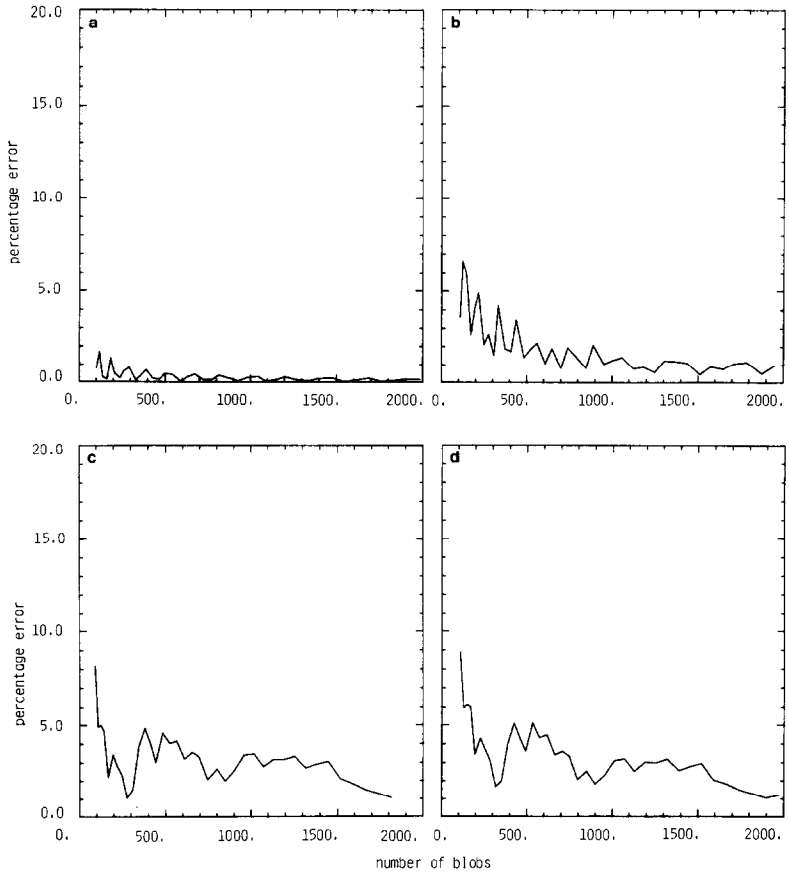

FIG. 3. (a) Initial error e_{startup} for uniformly distributed vortex blobs with averaged strength. (b) Initial error e_{startup} for uniformly distributed vortex blobs with unaveraged strength. (c) Initial error e_{startup} for randomly distributed vortex blobs with averaged strength. (d) Initial error e_{startup} for randomly distributed vortex blobs with unaveraged strength.

error by an amount of $N^{-1}(x^2 + y^2)$, N being the total number of blobs for a particular run. Hence the addition of one blob near the boundary of the support of the vorticity will produce a change of $1/N$.

Comparison of Figs. 3a–d clearly shows the superior accuracy of the uniform grid with blob strength given by (1), in approximating the initial angular moment. Figures 4a–d show the error $e(3T)$ (T is the period of the exact solution for the inviscid case) obtained when blobs with the various initial conditions were moved by the random walk algorithm ($R = 1250$). There is a correlation between the errors shown in Figs. 3c, d and Figs. 4c, d. This shows that for the methods where the blobs have a random distribution at $t = 0$ the initial error, as measured by e_{startup}, dominates the error produced by the random walks. Figures 3b and 4b show that the phenomenon is less for a uniform initial distribution. Finally the averaging gives

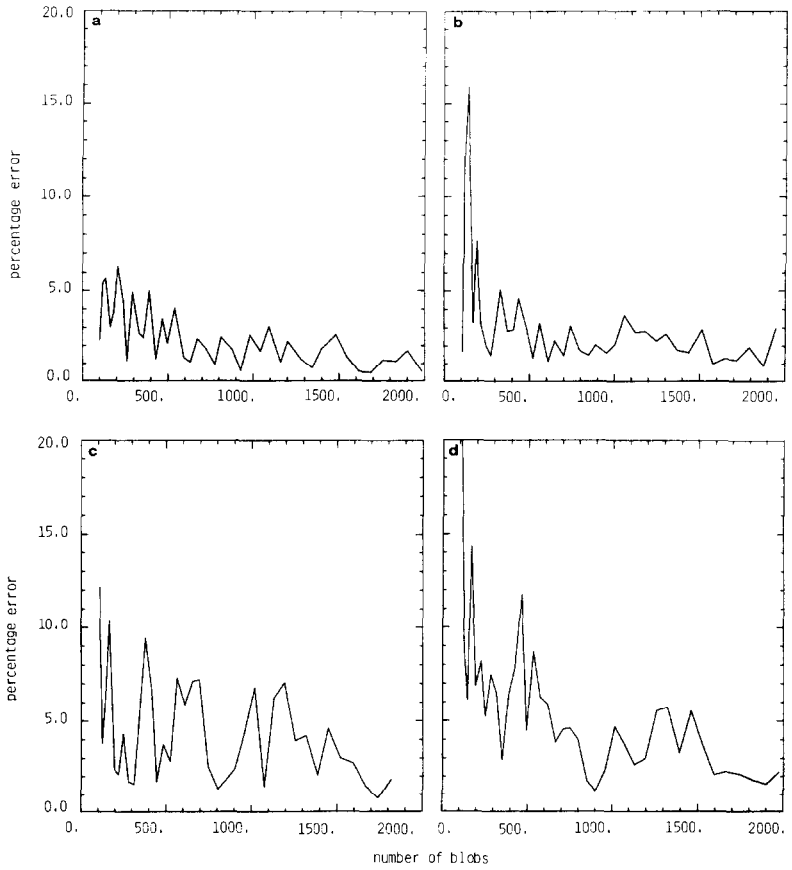

FIG. 4. (a) Error $e(3T)$ for uniformily distributed vortex blobs with averaged strength evolved by random walks with $R = 1250$. (b) Error $e(3T)$ for uniformily distributed vortex blobs with unaveraged strength evolved by random walks with $R = 1250$. (c) Error $e(3T)$ for randomly distributed vortex blobs with averaged strength evolved by by random walks with $R = 1250$. (d) Error $e(3T)$ for randomly distributed vortex blobs with unaveraged strength evolved by by random walks with $R = 1250$.

the best results. This is expected since the initial data is not smooth. In the remainder of this paper we will use uniform grid and averaging.

For runs with Reynolds number greater than 20000, it became evident that even with the use of a uniform grid and averaging that startup errors were distorting the errors produced by the interaction of the two fractional steps of our method (see Table II). To eliminate the effect of the startup error we have introduced the functional e_1.

We observed that practically any choice of cutoff function and cutoff δ produced similar results. This seems to be due to our method of error measurement. If the vortex blobs have a distribution which is radially symmetrical, then the velocity

TABLE II

Comparison of Error Measurements

	Random Vortex Method			Random Walk Algorithm		
Reynolds Number	$e(T)$	$e_1(T)$	$e_{MS}(T)$	$e(T)$	$e_1(T)$	$e_{MS}(T)$
80000	0.408	0.310	324.	0.602	0.108	478.
20000	0.196	0.536	39.1	0.516	0.193	103.
5000	0.453	1.08	22.5	0.362	0.354	18.0
1250	1.46	2.00	18.2	0.434	0.634	5.39

Note. Measurements in %, for runs with 500 blobs and with $e_{\text{startup}} = 0.239\%$.

given by (2.12), for any cutoff, has no mean radial component. As our modes of error measurement only detect changes in radial positions (L depends on distance of blobs from origin), we cannot accurately compare different cutoff functions. On the other hand, we need a cutoff function if $R \neq \infty$, since the random walk algorithm allows blobs to become arbitrarily close. Hence if no cutoffs were used, the interactions would be unstable. We chose Chorin's cutoff function with $\delta = 2h$.

From (3.13), for a fixed number of blobs we expect that the error would be $O(R^{-1/2})$, at least for pure diffusion. Table II tabulates our results as we increased Reynolds number from 1250 to 80000. With 500 blobs, note that the errors for the random vortex method are larger than those obtained for random walks, but that the rates of convergence as $R \to \infty$ are similar. These results verify the estimates established in Section 3 for the random walk simulation.

How does the number of vortex blobs affect the accuracy. For $R = 1250$,

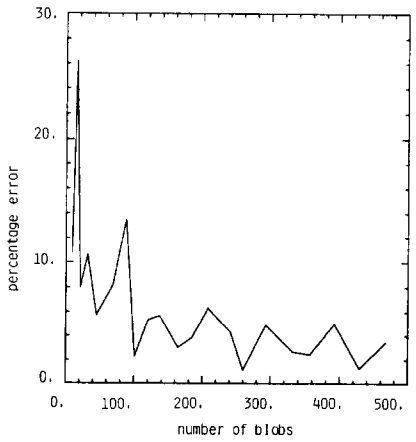

FIG. 5. Error $e(3T)$ for uniformily distributed vortex blobs with averaged strength evolved by random walks with $R = 1250$.

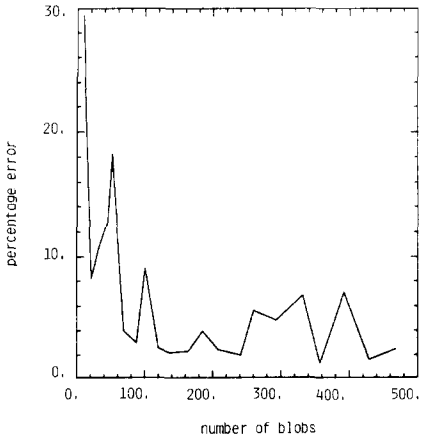

FIG. 6. Error $e(3T)$ for uniformily distributed vortex blobs with averaged strength evolved by random vortex method with $R = 1250$.

Figs. 4a, 5, and 6 and Table III contain the relevant data. First note that a comparison of the plots in Figs. 5 and 6 indicates that the random vortex method and the random walk algorithm produce qualitatively similar results. To study the behavior of the results as $N \to \infty$, we calculated the functions of the form cN^{α} which best fit the data plotted in the Figs. 4a–d and Fig. 6. In particular, we calculated the least square estimator for the data sets $\log(e)$ vs. $\log(N)$. In addition, we determined the function of best fit for the points of local maxima in those figures. These results along with the appropriate correlation coefficients are tabulated in Table III. We see that Figs. 4a, b, d, and Fig. 6 all demonstrate behavior (α close to -0.5) which is consistent with the error being $O(N^{-1/2})$, as $N \to \infty$.

TABLE III

Least Square Estimator

	Data Sets					
	All (N, e)			(N, e) Local Maxima		
Figure	α	β	Corr. Coef.	α	β	Corr. Coef.
4a	−0.533	4.09	−0.667	−0.535	4.59	−0.905
4b	−0.418	3.59	−0.566	−0.591	5.17	−0.838
4c	−0.280	3.06	−0.340	−0.359	4.02	−0.634
4d	−0.572	5.19	−0.745	−0.617	5.85	−0.883
6	−0.563	4.27	−0.741	−0.474	4.35	−0.818

Note. $\log(e) = \alpha \log(N) + \beta$, for data e vs. N contained in Figs. 4a–d and 6.

5. Conclusion

We have shown that the choice of initial conditions are important in the random vortex method. With our initial vorticity a uniform grid initial distribution of blobs with the strength of the blobs obtained by averaging give the best results. The dependence on the initial conditions is also found in the vortex method for smooth initial data. Here though, the choice of cutoff function is also important (Beale and Madja [4] and Perlman [14]).

The time evolution of the errors obtained for the random vortex method is similar to the corresponding result for the random walk algorithm. This is a favorable outcome. We could not expect the fractional step algorithm to produce less error than its individual parts, but it was conceivable that the errors would be much larger.

Finally, our results indicate that the error e_1 for the random vortex method is $O(N^{-1/2}R^{-1/2})$. Hence, for a fixed number of vortex blobs, the accuracy of the method increases as the Reynolds number increases.

Acknowledgments

I would like to thank Professor A. Chorin for suggesting this problem and for his many helpful suggestions during the writing of this paper. In addition I thank Professor O. Hald for his many insightful comments.

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